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Gaussian fields and random flow

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The high-frequency component of the random solution of a model problem is shown to be statistically orthogonal to the Gaussian component. This is shown to be a consequence of the existence of an equilibrium range. It is concluded that random flow fields can be viewed as being approximately Gaussian only in a very special sense and, in particular, that Wiener–Hermite expansions can provide a useful description only of large-scale hydrodynamical phenomena.

1. Introduction

It is a well-known experimental fact that the velocity distribution of a homogeneous turbulent flow appears to be nearly Gaussian, but it is also well known that approximation methods which attempted to exploit this fact have ended in failure. This situation is in sharp contrast to the one which prevails in the kinetic theory of gases, where Gram–Charlier expansions have proved quite useful (see, for example, Chorin 1972). It is the purpose of this paper to provide a pithy resolution of this apparent paradox by showing that, contrary to appearances, random flow is not in fact nearly Gaussian, except in a very special sense which will be made explicit.

The problem at hand can be formulated as follows. Consider the differential equation

$$v_t = Qv, \tag{1}$$

where t is the time and Q is a nonlinear differential operator, with (1) representing either a model equation or the Navier–Stokes equations themselves. Let v^0 be a homogeneous random field, i.e. a random function of position x whose statistical properties are independent of x. Furthermore, let v^0 be Gaussian. (For precise definitions and analysis, see Gelfand & Vilenkin 1964, p. 237; Doob 1953, p. 71.) Let each realization of v^0 be taken as an initial state for (1) and allowed to evolve as the equation dictates. At a time $t > 0$ we obtain a random field v^t which is homogeneous if the coefficients of Q are independent of x, and whose statistical properties we should like to analyse. It is known that, even though v^0 is Gaussian, v^t is not. One can readily show that if the field remains Gaussian no energy transfer between modes can occur; it is, however, known that in solutions of the Navier–Stokes equations such transfer does occur. It has also been shown by Hopf (1952) that a random field containing an inertial range of frequencies, which is generally believed to exist for the problems at hand, is in an essential way distinct from a Gaussian field. Our purpose is to show the drastic extent of this distinction.

In the next section we present some of the tools we shall require, in particular the moving average and Wiener–Hermite representations of random fields. In §3 we discuss the model equation

$$v_t + vv_x = 0 \tag{2}$$

and the Burgers equation

$$v_t + vv_x - R^{-1}v_{xx} = 0, \tag{3}$$

where R is a large parameter, and show that the first term, i.e., the Gaussian term, in the Wiener–Hermite expansion of their solution, is orthogonal to the high-frequency component of that solution. In §4 we explain these results in terms of turbulence theory, and in particular contrast the notation of an equilibrium range with ideas known to be valid in classical statistical mechanics of a large but finite system of particles. In §5 we indicate how to generalize these results to the Navier–Stokes equations and how to use them to practical advantage.

2. Moving average and Wiener–Hermite representations of random fields

Let $\phi(x)$ be a random field and assume that it satisfies the following conditions.

(i) It has zero mean, i.e.

$$E[\phi(x)] = 0 \quad \text{for all } x,$$

where $E[\eta]$ describes the expected value of the random variable η.

(ii) It has locally finite energy

$$E[|\phi(x)|^2] < \infty.$$

(iii) It is homogeneous in the wide sense, i.e. the expression

$$B(r) = E[\phi(x)\,\phi(x+r)]$$

is a function of r alone,

(iv) $B(r)$ is continuous at $r = 0$.

Under these conditions, $\phi(x)$ has a spectral representation

$$\phi(x) = \int \exp(ikx)\,\mu(dk), \tag{4}$$

where μ is a random measure (Gelfand & Vilenkin 1964, p. 268; Doob 1953, p. 527). Furthermore,

$$B(r) = \int \exp(ikr)\,dF(k), \tag{5}$$

where $dF(k) = E[|\mu(dk)|^2]$ and $F(k)$ is the spectral distribution function of ϕ. The representations (4) and (5) are well known and commonly used in the theory of turbulence (see, for example, Batchelor 1960, p. 28).

A random field is said to have orthogonal increments if, whenever A and B are disjoint measurable sets on the x axis and

$$\eta_A = \int_A d\eta(x), \quad \eta_B = \int_B d\eta(x),$$

we have

$$E[\eta_A \eta_B] = 0.$$

Assume that ϕ has the following additional property.

(v) F is absolutely continuous, i.e. there exists an integrable function F' such that

$$dF(k) = F'(k)\,dk;$$

in other words, the energy density associated with ϕ in wavenumber space is finite. The relevance of this assumption to hydrodynamics is discussed, for example, in Batchelor (1960, pp. 25, 85).

We have the following theorem.

If, and only if, $F(k)$ is absolutely continuous, ϕ is a process of moving averages, i.e. there exists a fixed function $f^*(x)$ and a field $\eta(x)$ with orthogonal increments such that

$$\phi(x) = \int f^*(x-s)\,d\eta(s), \tag{6}$$

in the root-mean-square sense, where $d\eta$ is normalized by $E[|d\eta(s)|^2] = ds$.

The proof of the theorem is given, for example, in Doob (1953, p. 532). It is based on the remark that, if F is absolutely continuous, (4) can be written as

$$\phi(x) = \int \exp(ikx) f(k)\,d\eta^*(k),$$

where $f(k) = (F'(k))^{\frac{1}{2}}$ and η^* has orthogonal increments and is normalized by

$$E[|d\eta^*(k)|^2] = dk.$$

The Fourier transform of η^* is η, and the desired result follows by application of Parseval's identity. The theorem states that ϕ is a linear combination, with random coefficients, of functions obtained from a fixed function by translation. Furthermore, since $f = F'^{\frac{1}{2}}$, we have

$$B(r) = \int |f|^2 \exp(ikr)\,dk,$$

so that in particular

$$E[|\phi|^2] = \int |f|^2\,dk. \tag{7}$$

Thus, the spectrum of ϕ is contained in each one of the elements whose sum makes up ϕ.

In the particular case of a Gaussian field ϕ, η and η^* are Gaussian fields with orthogonal (and thus independent) increments. However, the field η in (6) is arbitrary to a very large extent; fields which are neither Gaussian nor close to being Gaussian have representations of the form (6).

Below we shall be interested not only in representing given random fields, but also in studying their evolution in time subject to a given differential equation. In general, as ϕ evolves, so do both f^* and η. It would be convenient to have a representation of ϕ in terms of an unchanging random field, and then have to deal solely with the variation in time of sure (non-random) functions such as f^*. Such a representation is given by the Wiener–Hermite expansion (Wiener 1958, p. 16; Cameron & Martin 1947; Meecham & Jeng 1968). Let $\xi(x)$ be a Gaussian process with orthogonal increments, normalized by $E[|d\xi|^2] = dx$; ξ is uniquely determined up to an immaterial constant in the normalization condition. Let

$$H_n(d\xi(s_1), d\xi(s_2), \ldots, d\xi(s_n)) = H_n(d\xi)$$

be the Hermite polynomial functional of ξ (further defined below); let $\phi(x)$ be a random field with the following additional property: for each x, $\phi(x)$ is a functional of ξ which is square-integrable with respect to the Wiener measure; then we have

$$\phi(x) = \sum_{n=0}^{\infty} \int \dots \int K_n(x-s_1, x-s_2, \dots, x-s_n)\, H_n(d\xi). \tag{8}$$

The kernels K_n are square-integrable sure functions, and the randomness is expressed by the presence of the orthogonal functionals $H_n(d\xi)$. One could hope to substitute the expansion into a differential equation (1), and use the orthogonality of the $H_n(d\xi)$ to obtain relations between the kernels K_n. This is in fact the basis of the method of Meecham & Jeng (1968).

Note that, if ϕ is Gaussian, the representations (6) and (8) coincide; $\eta(s)$ is Gaussian, while (8) contains a single, Gaussian term. The terms $n = 2, \dots$, represent the non-Gaussian part of ϕ.

3. Wiener–Hermite expansion of solutions of model problems

Consider first the model problem (2):

$$v_t + vv_x = 0, \quad v(x, 0) \text{ given.}$$

Some of its relevant properties are the following: except for very exceptional cases, shocks will develop in $v(x,t)$ in a finite time however smooth the data may be; the number of shocks will ultimately decrease through overtaking and absorption; if $v(x, 0)$ has compact support, one ends up with a single shock. The shocks are responsible for the decay of the solutions. The number of shocks per unit length of the x axis is finite. C_i, the intensity of the shock at x_i, depends on the data between x_+ and x_-, where x_+ and x_- are the intersections with the x axis of the characteristics entering the shock from the right and left. The existence of these characteristics is guaranteed by the entropy condition, which allows only compression shocks (see Glimm & Lax 1970).

Let $\hat{v}(k)$ be the Fourier transform of $v(x)$, whenever it is defined. An 'energy cascade' occurs in the solution of (2), i.e. 'modes' $\hat{v}(k)$ with ever increasing k are excited. This cascade is occasioned by the appearance of shocks. Since the Fourier transform of a shock of intensity C_i at x_i is $C_i e^{-x_i k} k^{-1}$, the energy density in the high wavenumber range is $O(k^{-2})$.

Let $q(x)$ be the function defined by

$$q(x) = \begin{cases} (a+x)/a, & -a < x \leqslant 0, \\ (x-a)/a, & 0 \leqslant x < a, \\ 0, & a \leqslant |x|. \end{cases}$$

$q(x)$ is discontinuous at $x = 0$ and has a support of length $2a$. Let $v(x,t)$ be the solution of (2). We have for all t

$$v(x,t) = u(x,t) + w(x,t), \tag{9}$$

where $u(x,t)$ is continuous and $w(x,t)$ has the form

$$w(x,t) = \Sigma - C_j q(x - x_j), \tag{10}$$

where the x_j are the locations of the shocks of v and the C_j are their intensities. For each x, the sum in (10) contains only a finite number of terms.

Consider the initial data as being a single realization of a Gaussian random field $v^0(x)$:

$$v^0(x) = \int f^*(x-s)\, d\xi(s), \quad d\xi(s) \text{ Gaussian.}$$

Each realization of v^t can be written in the form (10), and thus

$$v^t = u^t + w^t,$$

where each realization of u^t is continuous. We call w^t the discontinuous part of v^t. u^t and w^t are both homogeneous random fields satisfying conditions (i)–(v) above. We have the following theorem.

The Gaussian term in the Wiener–Hermite expansion of w^t is zero, i.e.,

$$K_1(s) = E[w^t(0)\, d\xi(s)] = 0. \tag{11}$$

Note that the theorem is clearly suggested by the contrast between the representation (6) applied to a Gaussian field, which describes the field as an infinite sum of identical objects, and the fact that w^t contains only a finite sum of identical objects in each realization. We shall prove the theorem by a method both concrete and laborious, with the purpose of not only proving the result, but also exhibiting in detail what it is that goes awry.

We begin by expressing the Wiener–Hermite expansions as a limit of Hermite functions of a discrete set of linear functions of ξ (Imamura, Meecham & Siegel 1965; Cameron & Martin 1947). Consider the interval $I = \{x \mid -X \leqslant x \leqslant X\}$; let N be an integer; divide I into $N+1$ subintervals

$$I_i = \{x \mid -X + (i-1)\, h \leqslant x \leqslant -X + ih\},$$

where $h = 2X/N$. Let ψ_i be the function

$$\psi_i = \begin{cases} 1, & x \in I_i, \\ 0, & x \notin I_i. \end{cases}$$

ψ_i is the characteristic function of I_i. Let

$$\xi_i = \int \psi_i d\xi,$$

where ξ is a Gaussian field with orthogonal increments; the integrals ξ_i exist as generalized Stieltjes integrals for almost all ξ, and ξ_i is a Gaussian random variable with zero mean and variance

$$E[\xi_i^2] = h.$$

Consider the family of functionals of ξ defined by

$$H_0^h(\xi) = 1, \quad H_1^h(\xi) = \sum_i K_1(x-s_i)\, \xi_i,$$

$$H_2^h(\xi) = \sum_i \sum_j K_2(x-s_i, x-s_j)\, [\xi_i \xi_j - \tfrac{1}{2} h \delta_{ij}],$$

etc., where the K_i are square-integrable functions of their arguments, δ_{ij} is the Kronecker delta and $E[H_p^h H_q^h] = \delta_{pq}$. The functionals H_n^h are our coarse-grained

Wiener–Hermite polynomials; h is the grain size and we define a coarse-grained Wiener–Hermite expansion to be the series

$$\sum_n a_n H_n^h(\xi), \quad a_n \text{ constants.} \tag{12}$$

The Cameron–Martin (1947) theorem guarantees that as $h \to 0$ and then $X \to \infty$ the limit of the series exists provided that

$$\sum_n \int |K_n(s_1, \ldots, s_n)|^2 \, ds_1, \ldots, ds_n < \infty;$$

such series span the space of random fields which are square-integrable with respect to the Wiener measure.

We now turn to the field w^t. By the theorem of the last section, it has a representation of the form

$$w^t = \int f^*(x-s) \, d\eta(s).$$

We have to prove that η has a finite number of jumps per unit length. It is simpler to write

$$w^t = \underset{h\to 0,\, X\to\infty}{\text{l.i.m.}} \sum f^*(x-s_i) \, \eta_i, \tag{13}$$

where l.i.m. denotes a limit in the mean, s_i belongs to the support of ψ_i, and η_i is the sum of the C_j corresponding to shocks whose locations fall in the support of ψ_i. Equation (13) is valid without further ado. The question now is whether (13) can be expressed as a limit of an expression such as (12). Let the variables η_i be functions of the variables ξ_i;

$$\eta_i = G(\xi_1, \ldots, \xi_N). \tag{14}$$

(All the ξ_i must appear because we have no reason to believe that the η_i are independent.)

Let us assume for a moment that

$$\eta_i = g(\xi_i), \tag{15}$$

i.e. η_i depends on a single variable ξ_i. It will be obvious that the argument below generalizes to the general case (14), but the assumption (15) reduces the amount of writing necessary.

Consider the unit interval $[0, 1]$. The number of shocks in this interval is finite for each realization of w^t. Thus, for every $0 \leqslant \epsilon_1 < 1$ there exists an integer $M = M(\epsilon_1)$ such that the probability of finding more than M shocks in $[0, 1]$ is less than ϵ_1. Therefore, for every $0 < \epsilon_2 \leqslant 1$, however small, there exists an N such that the probability of finding no shocks in the support of any one ψ_j is greater than $1-\epsilon_2$: simply pick $M = M(\frac{1}{2}\epsilon_2)$ and $h = 2X/N \leqslant \frac{1}{2}M$. If there are no shocks in the support of ψ_j, $\eta_j = \int \psi_j \, d\eta = 0$. Thus the probability of having $\eta_j = 0$ is $1-\epsilon$, with $0 \leqslant \epsilon \leqslant \epsilon_2$.

We now construct a random variable η_j having this property as a function of ξ_j. Divide the interval $[0, 1]$ into two disjoint sets S_1 and S_2, with S_1 of length $1-\epsilon$ and S_2 of length ϵ. For the sake of convenience, introduce the notation

$$e_h(x) = (2\pi h)^{-\frac{1}{2}} \exp(-x^2/2h),$$

$$E_{h,\epsilon}(x) = \int_{S_2(x)} e_h(z) \, dz,$$

where $S_2(x)$ is the intersection of $[-\infty, x]$ with the set $\{x | e_h(x) \in S_2\}$. Assume for a moment that except at $x = 0$ the probability distribution function of η_j,

$$Q(x) = \text{prob}\,(n_j \leqslant x),$$

is differentiable, and let

$$Q_\epsilon(x) = \begin{cases} Q(x) & (x < 0), \\ Q(x) - (1-\epsilon) & (x > 0). \end{cases}$$

Then we can set $\eta_j = g_h(\xi_j)$, where

$$g_h(x) = \begin{cases} 0 & (e_h(x) \in S_1), \\ Q_\epsilon^{-1}(E_{h,\epsilon}(x)) & (e_h(y) \in S_2). \end{cases}$$

If $Q(x)$ is not differentiable, $g_h(x)$ is given for $e_h(y) \in S_2$ by a more complicated formula; the argument below is not modified in an essential manner.

We have, by definition,

$$E[\eta_j^2] = \int g_h^2(x)\, e_h(x)\, dx = h;$$

let $q(x)$ be the characteristic function of $e_h^{-1}(S_2)$, i.e.

$$q(x) = \begin{cases} 0 & (e_h(x) \in S_1), \\ 1 & (e_h(x) \in S_2). \end{cases}$$

We have then

$$\begin{aligned} 0 < \int_{-\infty}^{+\infty} |g_h(x)| e_h(x)\, dx &= \int |g_h(x)|\, q(x)\, e_h(x)\, dx \\ &\leqslant \left(\int |g_h(x)|^2\, e_h(x)\, dx\right)^{\frac{1}{2}} \left(\int q(x)\, e_h(x)\, dx\right)^{\frac{1}{2}} \\ &= h^{\frac{1}{2}} \epsilon^{\frac{1}{2}}. \end{aligned}$$

Thus, the inner product of g_h with any bounded function tends to zero faster than $h^{\frac{1}{2}}$ as $h \to 0$; collecting terms, we see that the leading term in the Wiener–Hermite expansion of w^t is the limit of $N = O(h^{-1})$ variables, each with variance $h\epsilon$, with $\epsilon \to 0$; thus it tends to zero as $h \to 0$. Q.E.D.

Note that we cannot conclude from our argument that the following terms in the expansion vanish. However, it is clear that the number of terms required to represent the field may be large, and will increase as the mean separation between shocks increases. The high-frequency range is not a small perturbation of a Gaussian field. Furthermore, if one expands in Wiener–Hermite polynomials not the field itself but its Fourier transform (Meecham & Jeng 1968), the results should be decreasingly valid as the frequency k increases. All these phenomena have in fact been observed (Crow & Canavan 1970).

The conclusion applies not only to the solution of (2), but also to the solution of Burgers' equation (3). The solution of the latter converges to the solution of the former as $R \to \infty$, and thus whenever the former gives rise to a finite number of shocks the latter gives rise to a finite number of shear layers, and the theorem applies.

We shall see in the final section that the proof above provides in fact additional information; we shall see how to use the fact that if the grain size remains finite, i.e. if one refrains from passing to the limit $h \to 0$ in expressions (12) and (13), then the expansion of the resulting approximation to (12) in Wiener–Hermite polynomials can be expected to be non-trivial. Before explaining further, we need some further elaboration of the results of this section.

4. Equilibrium and Gaussian fields

The results of the previous section may appear surprising for the following reason: the Gaussian component of the flow is orthogonal to the high-frequency range, where statistical equilibrium is supposed to be reached (see below); by analogy with kinetic theory, one is tempted to equate equilibrium and Gaussianity; one is faced with an apparent paradox, which we shall now proceed to resolve.

The range of wavenumbers k which contain most of the energy of hydrodynamical flow can be viewed as a group, with characteristic velocity $v_{en} = (E[v^2])^{\frac{1}{2}}$, characteristic length k_{en}^{-1}, where k_{en} is a typical wavenumber in the group, and characteristic time $(k_{en} v_{en})^{-1}$. The characteristic time of their decay is $v_{en}/|dv_{en}/dt|$; these times are experimentally found to be comparable (Batchelor 1960, p. 104), and thus conditions are far removed from those prevailing in classical equilibrium statistical mechanics. On the other hand, experience suggests that at high frequency k the eddies have a characteristic time small in comparison with the overall decay time, and thus may be associated with 'degrees of freedom' in approximate statistical 'equilibrium'.

Thus, an attempt has been made in recent years to equate 'equilibrium' and Gaussianity. It is known that the inviscid equation (2) formally admits as an invariant solution a 'Gaussian equipartition ensemble', i.e. it leaves invariant a Gaussian field with orthogonal values. It has been posulated (see, for example, Orszag (1967) and the references therein) that this field is the equilibrium field, and that any other field irreversibly relaxes towards it. However, one must remember that this Gaussian field is only a formal solution of the equations; the proof of its invariance contains operations whose validity is not clear, and a typical realization contains an infinite number of δ-functions, to which the application of nonlinear differential operators is of doubtful validity. It is readily shown that relaxation towards this field does not in fact take place. Consider an initial field satisfying condition (ii) above, i.e. having locally finite energy. This property is preserved by the equation. On the other hand, the realizations of the field with orthogonal values have almost surely an infinite energy (Friedrichs & Shapiro *et al.* 1956, p. V–4).

Let us re-examine the assumption of statistical equilibrium. Let k_{eq} be a wavenumber in the equilibrium range and let v_{eq} be a typical amplitude of $\hat{v}(k)$ for k in the equilibrium range. Write

$$K = k_{eq}/k_{en}, \quad U = v_{eq}/v_{en}$$

(thus K is large and U is small). The characteristic time of v_{eq} is

$$(k_{eq} v_{eq})^{-1} = (KU)^{-1}(k_{en} v_{en})^{-1},$$

and thus the assumption of universal equilibrium reads

$$(KU)^{-1}\,|dv_{en}/dt| \ll k_{en}\,v_{en}^2. \tag{16}$$

We now seek solutions of (2) having this property. The Fourier transform of $v_t + vv_x = 0$ is

$$\hat{v}_t + ik \int \hat{v}(k')\,\hat{v}(k-k')\,dk' = 0, \tag{17}$$

where $\hat{v}(-k)$ is the complex conjugate of $\hat{v}(k)$. We first note that a random field each one of whose realizations is a stationary solution of the differential equation is certainly left invariant by the equation and thus represents an 'equilibrium'. A stationary solution can be obtained by setting $\hat{v}_t = 0$ in (17); this leads to the Fourier transform of

$$(v^2)_x = 0,$$
$$v^2 = C^2, \quad C = \text{constant},$$
$$v = \pm C,$$

where different signs may be assumed on different parts of the x axis. The entropy condition allows only compression shocks, and thus only one change of sign, from plus to minus, can occur, leading to the field whose realizations have the transform

$$\hat{v} = 2C\,e^{iak}k^{-1},$$

where C and a are random. This field is clearly not Gaussian.

Now note that any sum of a smooth flow and a finite number of realizations of the field above satisfies condition (16), which defines the equilibrium range. Let k be in the equilibrium range, with $v(k)$ the corresponding amplitude. $v(k)$ satisfies (17). Perform the change of variables

$$v^* = v/U, \quad k^* = k/K,$$

U and K being defined above. Substitution into (17) yields

$$(KU)^{-1}\hat{v}_t^* = ik^* \int v^*(k')\,v^*(k-k')\,dk'. \tag{18}$$

$(KU)^{-1}\,\hat{v}_t^*$ is of order $(KU)^{-1}\,(dv_{en}/dt)$. On the other hand, the integral on the right-hand side of (18) contains terms of orders k_{en}, v_{en} and v_{en}. By (16) we find

$$\lim_{k\to\infty} \hat{v}_t(k) = \lim_{k\to\infty} ik \int \hat{v}(k')\,\hat{v}(k-k')\,dk' = 0. \tag{19}$$

This is not unexpected, since the assumption of equilibrium states that the rate of change of high-frequency components is small in appropriate units. We now show that a sum of a smooth flow (with $\hat{v}(k) = o(k^{-1})$ for large k) and of a finite number of shocks (two for the sake of economy in notation) satisfies (19). Indeed, let

$$\hat{v}(k) = C_1 \exp(ia_1 k)\,k^{-1} + C_2 \exp(ia_2 k)\,k^{-1} + o(k^{-1}).$$

Then

$$\begin{aligned} ik \int v(k')\,v(k-k')\,dk' &= ik[C_1 \exp(ia_1 k) + C_2 \exp(ia_2 k)] \int (k')^{-1}(k-k')^{-1}\,dk' \\ &\quad + ikC_1C_2 \exp(ia_2 k) \int \exp[i(a_1-a_2)k']\,k'^{-1}(k-k')^{-1}\,dk' + o(k^{-1}) \\ &= ikC_1C_2 \exp(ia_2 k) \int \exp[i(a_1-a_2)k']\,(k')^{-1}(k-k')^{-1}\,dk' + o(k^{-1}). \end{aligned}$$

Set $k = Kk^*$ and $k' = Kk^{*\prime}$, then this expression becomes

$$2ik^*C_1C_2\exp(ia_2k^*K)\int\exp[i(a_1-a_2)k^{*\prime}K](k^{*\prime})^{-1}(k^{*\prime}-k^{*\prime})\,dk^{*\prime},$$

which tends to zero as k, and thus K, tend to ∞, because of the presence of the oscillating exponential. Furthermore, note that, as the separation $a_1 - a_2$ of the shocks increases, the limit is approached faster.

This discussion can now be linked with what we know about the solutions of (2). Arbitrary data give rise to shocks; thus an equilibrium range is formed. The shocks overtake and absorb each other, their number diminishes and their separation increases; thus the range of wavenumbers which participate in the equilibrium field increases. The flow relaxes to the non-Gaussian equilibrium field discussed above. The phenomena of shock formation, energy cascading, relaxation to equilibrium and decay are in fact identical. The set of flows having these features is a negligible subset of the set of fields having a Wiener–Hermite expansion.

The discussion of the equilibrium range for (2) applies equally to the inertial range of Burgers' equation (3). The inertial range is the part of the equilibrium range where viscous dissipation is negligible. It can be studied by analysing the behaviour of the Fourier transform of the solutions of (3) as $R \to \infty$ and $k \to \infty$ in this order. However, the solutions of (3) tend to the solutions of (2) in L_1, and thus the finite limit can be studied by setting $R^{-1} = 0$, reverting to the previous case.

A comparison with classical statistical mechanics of a finite but large number of particles may be illuminating. The equilibria studied in this section and classical statistical equilibria are conceptually totally distinct. The analogy between classical statistical mechanics and statistical fluid mechanics is not generally valid, since, while in the former the particles which are in motion are fixed in nature, in the latter the random velocity field is both the agent and the object of the motion; what is being moved by the velocity field is the velocity field itself. The theory of the equilibrium range presented in this section isolates those features of the flow which are relatively slow to change, and are thus the ephemeral particles of the flow. It is to them, and thus to the averaged variables η_i, that classical ideas, in particular Gaussian expansion methods may apply.

5. Generalizations and applications

The preceding discussion is worthwhile only if its conclusions can be generalized to the Navier–Stokes equations and lead to practical conclusions. We shall presently show that they do.

Consider first the case of incompressible flow in two space dimensions. We have the following facts.

(*a*) The vorticity field ζ associated with such a flow, if it satisfies the obvious vector analogues of conditions (i)–(v) above, has the form

$$\zeta(\mathbf{x}) = \iint \zeta_0(\mathbf{x}-\mathbf{s})\,d\eta(\mathbf{x}), \tag{20}$$

where $\mathbf{x}$ and $\mathbf{s}$ are two component vectors, η is a field with orthogonal increments and ζ_0 is a fixed vorticity field. Furthermore, if $\zeta(\mathbf{x})$ is isotropic as well as homogeneous, $\zeta_0(\mathbf{x}) = \zeta_0(|\mathbf{x}|)$, i.e. ζ_0 is a circular vortex.

(*b*) The steady inviscid equations have as a solution any circular vortex, as well as any infinite straight vortex sheet. A field each one of whose realizations is a circular vortex is an 'equilibrium' field; an arbitrary superposition of circular vortices satisfies condition (16) (Chorin 1970).

(*c*) It is a long-standing conjecture (Onsager 1949), well supported by numerical and experimental evidence, that two-dimensional flow evolves through the gradual consolidation of vortices.

(*d*) Approximation methods based on Gram–Charlier or Wiener–Hermite expansions have proved to be failures.

In the presence of these facts, one can conjecture that the situation is analogous to the one previously discussed. Relaxation to equilibrium proceeds through the gradual consolidation of vortices, tending to the 'equilibrium' discussed above. In any one realization, the number of vortices present per unit area is finite (there is an ambiguity in this statement, due to the difficulty of defining precisely an individual vortex other than a point vortex). The flow fails to be Gaussian exactly in the manner above.

One obvious way to put the above discussion to practical use is to represent the flow field not by means of difference approximations or Fourier coefficients, but rather as a sum of a finite number of randomly placed vorticity elements, each one of which has a Fourier transform with an inertial range. Such a method was successfully developed in Chorin (1973*b*).

A more intriguing approach, presently being tried, can be presented in two distinct but equivalent ways. Suppose that the conjecture of this section is valid; $d\eta$ is not Gaussian. Let ψ_{ij} be functions of support of area h^2, where h is the grain size; the ψ_{ij} are the analogues of the ψ_i above. Consider the stochastic integrals

$$\eta_{ij} = \iint \psi_{ij}\, d\eta$$

and the corresponding Gaussian integrals

$$\xi_{ij} = \iint \psi_{ij}\, d\xi,$$

where ξ is a field with Gaussian increments. By the discussion above, and according to the proof of the theorem regarding the Wiener–Hermite expansion of w^t, the random variables are nearly Gaussian if h is large enough. One can construct the coefficients in the expansion, and have a reasonable numerical description of the vorticity field. Such descriptions are known to be valid and useful in other contexts (Chorin 1973*a*).

On the other hand, one may take literally the statement that the equilibrium theory identifies the temporary particles of the flow, and expand the vorticity density in small cubes in Wiener–Hermite series (or expand its density function in Gram–Charlier series), using the methods of kinetic theory (Chorin 1972). This is identical in practice to what has been described in the preceding paragraph. The requirement that h be not too small is now easily understood: before one can expand a vorticity density in a small cube in a Gaussian series, one needs at least a few vortices in that cube.

The three-dimensional case can be seen to be analogous; the computational details will be presented elsewhere.

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